Indian Statistical Institute, Bangalore Centre
B.Math (Hons.) II Year, First Semester
Back Paper Examination - 2013-2014
Analysis III

Time: 3 Hours January 2, 2014

Maximum marks you can score is 40. This paper carries 44 marks.

1. State the following theorems
   (i) Weirstrass theorem for $C[a, b]$
   (ii) Weirstrass M-test
   (iii) Change of Variable formula
   (iv) Inverse function theorem
   (v) Implicit function theorem [10]

2. Let $Q : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be any C^1 function. Let a bounded open subset of R^2 have the form
$$G = \{(x, y) : a < x < b,\ f_1(x) < y < f_2(x)\},$$
$$= \{(x, y) : c < y < d,\ g_1(y) < x < g_2(y)\}$$
where $f_1, f_2 : [a, b] \longrightarrow R,\quad g_1, g_2 : [c, d] \longrightarrow R$ are suitable continuous functions satisfying $f_1(x) < f_2(x)$ for x in (a, b) and $g_1(y) < g_2(y)$ for y in (c, d). Prove Greens theorem namely
$$\iint_G \frac{\partial Q}{\partial x}\, dx \ \ dy = \int_{\partial G} Q\ dy$$
where ∂G is traced in the anticlockwise direction. [6]

3. Let $f : (0, 1) \times (0, 1) \longrightarrow \mathbb{R}$ be given by
$$f(x, y) \ = \ \frac{x^2 - y^2}{(x^2 + y^2)^2}$$
Show that both the iterated integrals
$$\int_0^1 \left[\int_0^1 f(x, y) dx \right] dy, \ \ \int_0^1 \left[\int_0^1 f(x, y) dy \right] dx$$
exist but are not equal. [6]

4. Let $k : (a, b) \longrightarrow R$ be any bounded continuous function. Let $a < a_n < b_n < b$ and $a_n \to a, \quad b_n \to b$.
   (i) Show that $\lim_{n\to\infty} \int_{a_n}^{b_n} k(t)dt$ exists.
   (ii) Show that the limit is independent of the sequences of a_n, b_n as long or the limits are a and b.

[2+2]

5. Let $f, h : \mathbb{R} \longrightarrow \mathbb{R}$ be $\mathcal{C}^1$ functions. Let V be the convex closed tetrahedron of $\mathbb{R}^3$ with vertices $0 = (0,0,0),\ A = (a,0,0),\ B = (0,b,0)$ and $C = (0,0,c)$ where $a > 0,\ b > 0,\ c > 0$. Let $\underset{\sim}{F} = (f,0,0)$ Or $(0,0,h)$. Let S be the boundary of V, with outward normal orientation $\underset{\sim}{n}$. Show that $\iint\limits_S \underset{\sim}{F} = \iiint\limits_V \operatorname{div} \underset{\sim}{F}$. [6]

6. Let $F(x,y,z) = (y^2, xy, xz)$. Let S be the portion of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ above the plane $z = \frac{c}{2}$. Here $a > 0,\ b > 0,\ c > 0$. Find $\iint\limits_S \operatorname{curl} F$. [4]

7. Let $n > 2, a_1 > 0,\ a_2 > 0, \cdots a_n > 0$. Let

$$S(a_1, a_2, \cdots a_n) = \{(y_1, y_2, \cdots y_n) : |\frac{y_i}{a_i}| + |\frac{y_n}{a_n}| \le 1 \text{ for each } i = 1, 2, \cdots n-1\}$$

Calculate $\int\limits_{S(a_1,\ a_2,\ \cdots a_n)} dy$ [5]

8. Let $f_1, f_2, \cdots, f_n, \cdots$ be a sequence of real valued $\mathcal{C}^1$ functions on $[a,b]$. Let $h, g : [a,b] \longrightarrow R$ be continuous functions. If $f_n \longrightarrow g$ and $f_n' \longrightarrow h$ uniformly on $[a,b]$, then show that g is differentiable and $g' = h$. [3]